
QUADRATIC MODELING – PART I

Whether you're trying to maximize your profits in a factory, or trying to design a cattle corral of maximum area with a given amount of fence, much of economics and engineering involves making the most of limited resources. The vertex of a parabola will allow us to solve such optimization problems.



□ *SHORTCUT TO THE VERTEX*

Finding the vertex of a parabola by averaging the x -coordinates of the x -intercepts of the parabola is the same time-consuming process over and over again. Perhaps we can find the vertex of a generic parabola, yielding a simple formula that can find the vertex of any parabola quickly.

Let's start with a generic parabola (opening up or down) in standard form:

$$y = ax^2 + bx + c \quad [\text{opens up if } a > 0 \text{ and down if } a < 0]$$

Our job now is to find the x -intercepts of this parabola. To find any x -intercepts, we set y to 0 and solve for x :

$$0 = ax^2 + bx + c$$

Let's first turn the equation around so it's in standard form:

$$ax^2 + bx + c = 0$$

How do we solve this equation for x ? Actually, it's quite simple. The solutions of this equation are precisely the solutions provided by the Quadratic Formula. Thus, the two solutions of this equation are

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

These two “numbers” are the x -coordinates of the x -intercepts of the parabola. The next step is to find the average of these two numbers; this is done by adding them together and dividing by 2:

$$\begin{aligned} & \frac{\frac{-b + \sqrt{b^2 - 4ac}}{2a} + \frac{-b - \sqrt{b^2 - 4ac}}{2a}}{2} \\ &= \frac{\frac{-b + \sqrt{b^2 - 4ac} - b - \sqrt{b^2 - 4ac}}{2a}}{2} \quad (\text{common denominator}) \\ &= \frac{\frac{-2b}{2a}}{2} \quad (\text{the radicals sum to 0}) \\ &= \frac{\frac{-b}{a}}{2} \quad (\text{reduce the top fraction}) \\ &= \frac{-b}{a} \cdot \frac{1}{2} \quad (\text{multiply by reciprocal}) \\ &= \frac{-b}{2a} \end{aligned}$$

We've proved a theorem:

The x -coordinate of the **vertex** of the parabola

$$y = ax^2 + bx + c$$

is given by the formula

$$x = \frac{-b}{2a}$$

To find the y -coordinate of the vertex, just plug the x -coordinate of the vertex into the parabola formula.

For example, let's find the vertex of the parabola $y = -3x^2 - 24x - 42$. For this parabola, $a = -3$ and $b = -24$ (we don't need the value of c for this problem). So the x -coordinate of the vertex is

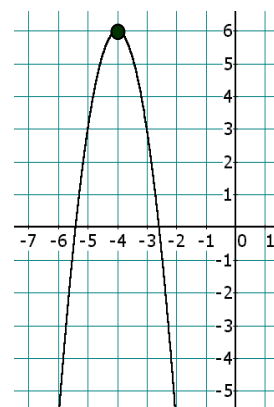
$$x = \frac{-b}{2a} = \frac{-(-24)}{2(-3)} = \frac{24}{-6} = -4$$

Now plug $x = -4$ into the parabola formula to find the y -coordinate of the vertex:

$$\begin{aligned} y &= -3x^2 - 24x - 42 = -3(-4)^2 - 24(-4) - 42 \\ &= -3(16) - 24(-4) - 42 = -48 + 96 - 42 = 6 \end{aligned}$$

Conclusion: The vertex of the parabola

$$y = -3x^2 - 24x - 42 \text{ is } (-4, 6).$$



Here's a good way to remember this vertex formula: Take a look at the Quadratic Formula, cross out the radical, and what remains is the x -coordinate of the vertex.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

(Note: In the original image, the $\pm \sqrt{b^2 - 4ac}$ part is crossed out with blue lines, leaving $x = \frac{-b}{2a}$.)

Homework

1. Use the formula $x = \frac{-b}{2a}$ to find the **vertex** of each parabola:

a. $y = x^2 + 2x - 48$

b. $y = x^2 + 10x + 25$

c. $y = 2x^2 + 8x + 5$

d. $y = 3x^2 - 6x + 4$

e. $y = -5x^2 + 40x - 20$

f. $y = -3x^2 - 16$

2. Use the formula $x = \frac{-b}{2a}$ to find the **vertex** of each parabola:

a. $y = -2x^2 + 5x - 7$

b. $y = 7x^2 - x - 1$

c. $y = 3x^2 - 3x - 2$

d. $y = 7x^2 + 10$

e. $y = 4x^2 + 7x$

f. $y = -5x^2 - 14x + 10$

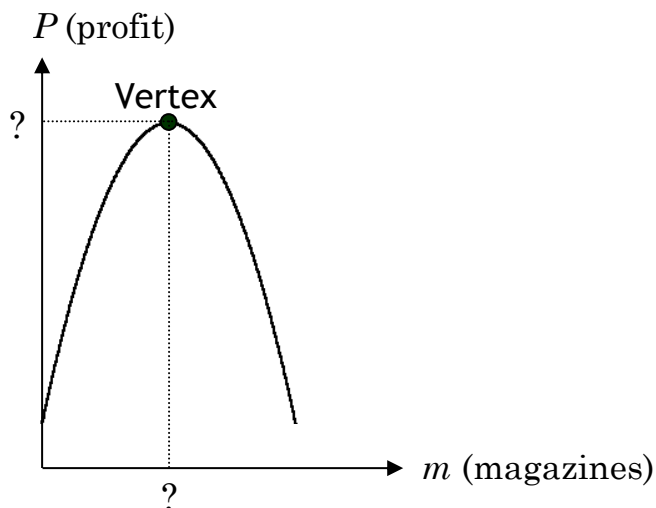
□ OPTIMIZATION PROBLEMS

EXAMPLE 1: A publishing company found that the profit they make on their latest magazine is given by the formula

$$P = -2m^2 + 680m + 10200$$

where P is profit and m is the number of magazines sold. Find the number of magazines that the company needs to produce in order to maximize their profit. Also determine the maximum profit.

Solution: Wow, what a problem! It might help to realize that the variable m is playing the role of the variable x , while the P is playing the role of y . When viewed this way, the profit formula is simply a parabola that opens down (since $-2 < 0$). And where is the maximum point on a parabola that opens down? At its vertex, of course.



Once we find the vertex of the given parabola formula, we can find the number of magazines needed to maximize the profit, and then the dollar amount of the maximum profit itself.

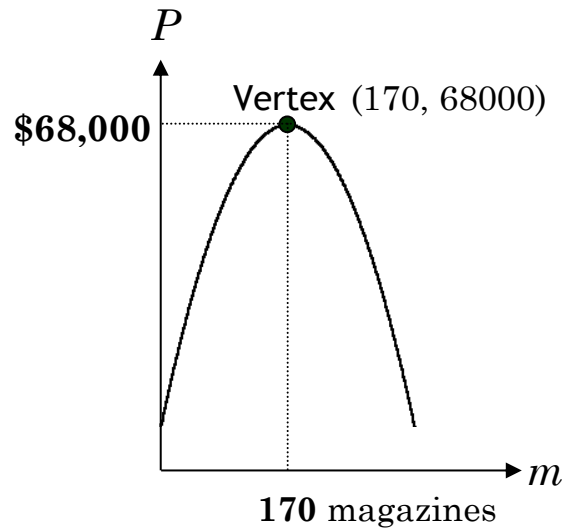
From the given parabola formula $P = -2m^2 + 680m + 10200$, we see that $a = -2$ and $b = 680$. Plugging these two values into our vertex formula gives us the m -coordinate of the vertex:

$$m = \frac{-b}{2a} = \frac{-680}{2(-2)} = \frac{-680}{-4} = \mathbf{170}$$

And since the m -axis is the horizontal axis (the magazine axis), we conclude that the number 170 represents magazines. That is, 170 magazines will produce the maximum profit. To calculate that profit, we merely let $m = 170$ in the profit formula:

$$\begin{aligned}
 P &= -2m^2 + 680m + 10200 && \text{(the profit formula)} \\
 &= -2(\mathbf{170})^2 + 680(\mathbf{170}) + 10200 && (m = 170) \\
 &= -2(28900) + 680(170) + 10200 && \text{(exponents first)} \\
 &= -57800 + 115600 + 10200 && \text{(multiplications second)} \\
 &= \mathbf{68000} && \text{(additions last)}
 \end{aligned}$$

The vertex of the parabola is therefore the point (170, 68000).



In conclusion,
producing **170**
magazines will yield
the company a
maximum profit of
\$68,000.

EXAMPLE 2: A company found that the cost for manufacturing a surfboard is given by the formula

$$C = 3s^2 - 270s + 6800$$

where C is the cost in dollars and s is the number of surfboards produced. Find the number of boards the company needs to produce in order to minimize their cost. What is the minimum cost?

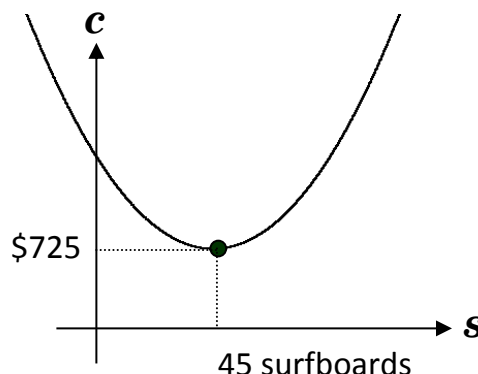
Solution: This problem is very similar to Example 1, except here we are to find the minimum of something, not the maximum. The graph of the cost formula is a parabola opening up (since $3 > 0$), so its vertex will be at the bottom of the parabola, and, as before, the vertex tells us everything we need to know.

Since $a = 3$ and $b = -270$, the s -coordinate of the vertex is

$$s = \frac{-b}{2a} = \frac{-(-270)}{2(3)} = \frac{270}{6} = 45$$

Plugging this value of s into the cost formula yields $C = 725$. Therefore, the vertex of the parabola is **(45, 725)**.

What does all this mean? Geometrically, the point (45, 725) is at the bottom of the parabola. Therefore,



producing **45** surfboards will result in a minimum cost of **\$725**.

Homework

3. A hard drive company found that the profit they make on their hard drive is given by the formula

$$P = -4h^2 + 200h + 1000,$$

where P is profit and h is the number of hard drives sold. Find the number of hard drives that the company needs to produce in order to maximize their profit. Also determine the maximum profit.

4. A company found that the cost for manufacturing a flash drive is given by the formula

$$C = 7d^2 - 140d + 745,$$

where C is the cost in dollars and d is the number of flash drives produced. Find the number of flash drives the company needs to produce in order to minimize their cost. What is the minimum cost?

5. The path of a baseball follows the parabola

$$y = -12x^2 + 456x + 100$$

At what value of x does the baseball reach its highest point? How high above the x -axis is the highest point?

6. The profit, P , is given by the formula

$$P = -10c^2 + 800c + 10000$$

Find the number of calculators, c , which would maximize the profit, and determine the maximum profit.

7. The cost, C , for manufacturing w widgets is given by

$$C = 7w^2 - 112w + 5000$$

How many widgets will minimize the cost, and what is the cost?

8. The path of a catapulted math teacher follows the parabola $y = -12x^2 + 312x + 250$. Find the point in the x - y plane where the teacher reaches his highest point.

9. A certain mountain has the approximate shape of the parabola $y = -13x^2 + 390x + 200$. Find the coordinates of the mountain peak.

10. A certain valley has the approximate shape of the parabola $y = 7.5x^2 - 1485x - 400$. Find the coordinates of the lowest point of the valley.

Review Problems

11. Using the short formula from this chapter, find the vertex of the parabola

$$y = -5x^2 + 70x - 90$$

12. The profit is given by the formula

$$P = -5b^2 + 750b + 10000$$

where P is profit and b is the number of books published. Find the number of books which would maximize the profit, and determine the maximum profit.

13. The cost, C , for manufacturing t trumpets is given by

$$C = 6t^2 - 276t + 5000$$

How many trumpets will minimize the cost, and what is the cost?

14. The path of a football follows the parabola $y = -7x^2 + 126x + 250$. Find the point in the x - y plane where the football reaches its highest point.

Solutions

- | | | |
|---|---|--|
| 1. a. $(-1, -49)$ | b. $(-5, 0)$ | c. $(-2, -3)$ |
| d. $(1, 1)$ | e. $(4, 60)$ | f. $(0, -16)$ |
| 2. a. $\left(\frac{5}{4}, -\frac{31}{8}\right)$ | b. $\left(\frac{1}{14}, -\frac{29}{8}\right)$ | c. $\left(\frac{1}{2}, -\frac{11}{4}\right)$ |

d. $(0, 10)$ e. $\left(-\frac{7}{8}, -\frac{49}{16}\right)$ f. $\left(-\frac{7}{5}, \frac{99}{5}\right)$

- 3. 25 hard drives; maximum profit = \$3,500
- 4. 10 flash drives; minimum cost = \$45
- 5. $x = 19$; height = 4,432
- 6. 40 calculators; maximum profit = \$26,000
- 7. 8 widgets; minimum cost = \$4552
- 8. $(13, 2278)$ 9. $(15, 3125)$
- 10. $(99, -73907.5)$ 11. $V(7, 155)$
- 12. 75 books; \$38,125 13. 23 trumpets; \$1826
- 14. $(9, 817)$

“The purpose of learning
is growth, and our
minds, unlike our
bodies, can continue
growing as we
continue to live.”



– Mortimer Adler